On $\mathbb{Z}$-graded loop Lie algebras, loop groups, and Toda equations

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Abstract

Toda equations associated with twisted loop groups are considered. Such equations are specified by $\mathbb{Z}$-gradations of the corresponding twisted loop Lie algebras. The classification of Toda equations related to twisted loop Lie algebras with integrable $\mathbb{Z}$-gradations is discussed.

1 Introduction

A Toda equation is a matrix differential equation of a special form equivalent to a set of second order nonlinear differential equations. It is associated to a Lie group and is specified by a $\mathbb{Z}$-gradation of the corresponding Lie algebra [1–3]. Of certain interest are also higher grading [4–7] and multi-dimensional [8, 9] generalizations of Toda systems. See also [10, 11], where affine Toda systems were treated as two-loop WZNW gauged models with potential terms.

When the Lie groups are from the list of the finite dimensional complex classical Lie groups, a group-algebraic classification of Toda equations associated with them was performed in the papers [12–14] where they were explicitly written in convenient block matrix forms induced by the $\mathbb{Z}$-gradations.

The group-algebraic and the differential-geometry properties of Toda systems and their physical implications are essentially different depending on what Lie group, finite or infinite dimensional, they are associated to. An instructive master example can be provided by two simplest cases of Toda systems, the Liouville and the sine-Gordon equations, with their well-studied drastic differences. In general, for the case of loop Lie groups one deals with infinite dimensional manifolds [15], which may give rise to additional problems compared to the finite dimensional case. Say, when one deals with an arbitrary $\mathbb{Z}$-gradation of a loop Lie algebra, one should take care of possible divergences in infinite series of grading components. A careful examination of loop groups of complex simple Lie groups and the corresponding $\mathbb{Z}$-graded loop Lie algebras was undertaken in [16]. In particular, a useful notion of integrable $\mathbb{Z}$-gradations was introduced there. On the basis of that consideration, Toda equations associated

with loop groups of complex classical Lie groups were explicitly described in a subsequent paper [17]. Here we review the papers [16, 17] skipping the proofs and concentrating mainly on the investigation logics.

2 Toda equations associated with loop Lie groups

2.1 General definition of Toda equation

Here $\mathcal{M}$ denotes either the Euclidean plane $\mathbb{R}^2$ or the complex line $\mathbb{C}$. We denote the standard coordinates on $\mathbb{R}^2$ by z^- and z^+. The same notation is used for the standard complex coordinate on $\mathbb{C}$ and its complex conjugate, $z = z^-$ and $\bar{z} = z^+$ respectively. As usual one writes $\partial_- = \partial/\partial z^-$ and $\partial_+ = \partial/\partial z^+$.

Recall that a Lie algebra $\mathfrak{G}$ is said to be *$\mathbb{Z}$-graded* if there is given a representation of $\mathfrak{G}$ in the form of the direct sum of subspaces $\mathfrak{G}_k$ such that

$$[\mathfrak{G}_k, \mathfrak{G}_l] \subset \mathfrak{G}_{k+l}$$

for any $k, l \in \mathbb{Z}$. This means that any element ξ of $\mathfrak{G}$ can be uniquely represented as

$$\xi = \sum_{k \in \mathbb{Z}} \xi_k,$$

where $\xi_k \in \mathfrak{G}_k$ for each $k \in \mathbb{Z}$. In the case when $\mathfrak{G}$ is an infinite dimensional Lie algebra we assume that it is endowed with the structure of a topological vector space and that the above series converges absolutely.

Let $\mathcal{G}$ be a Lie group with its Lie algebra $\mathfrak{G}$ supplied with a $\mathbb{Z}$-gradation. Assume that for some positive integer L the grading subspaces $\mathfrak{G}_{-k}$ and $\mathfrak{G}_k$ are trivial whenever $0 < k < L$. According to the definition of a $\mathbb{Z}$-gradation, its zero-grade subspace $\mathfrak{G}_0$ is a subalgebra of $\mathfrak{G}$, and one denotes by $\mathcal{G}_0$ the connected Lie subgroup of $\mathcal{G}$ corresponding to this subalgebra. The *Toda equation* associated with the Lie group $\mathcal{G}$ is a second order nonlinear matrix differential equation for a smooth mapping Ξ from $\mathcal{M}$ to $\mathcal{G}_0$, explicitly of the form[1]

$$\partial_+(\Xi^{-1}\partial_-\Xi) = [\mathcal{F}_-, \Xi^{-1}\mathcal{F}_+\Xi], \tag{1}$$

see, in particular, the books [1, 3]. In this equation, $\mathcal{F}_-$ is some fixed mapping from $\mathcal{M}$ to $\mathfrak{G}_{-L}$ and $\mathcal{F}_+$ is some fixed mapping from $\mathcal{M}$ to $\mathfrak{G}_{+L}$, which satisfy the conditions

$$\partial_+\mathcal{F}_- = 0, \qquad \partial_-\mathcal{F}_+ = 0. \tag{2}$$

When the Lie group $\mathcal{G}_0$ is abelian, one says that the corresponding Toda equation is *abelian*, otherwise one deals with a *non-abelian* Toda equation. Remember also that the Toda equation can be obtained within the differential-geometry framework, from the zero curvature condition on a flat connection in the trivial principal fiber bundle $\mathcal{M} \times \mathcal{G} \to \mathcal{M}$ imposing certain grading and gauge conditions [3].

The authors of the paper [10] consider equations of the form (1) for the case when L does not satisfy the condition that the subspaces $\mathfrak{G}_{-k}$ and $\mathfrak{G}_{+k}$ are trivial whenever

[1] We assume for simplicity that $\mathcal{G}$ is a subgroup of the group formed by invertible elements of some unital algebra $\mathcal{A}$. In this case $\mathfrak{G}$ can be considered as a subalgebra of the Lie algebra associated with $\mathcal{A}$. Actually one can generalize our consideration to the case of an arbitrary Lie group $\mathcal{G}$.

$0 < k < L$. Actually their assumption does not lead to new Toda equations. Indeed, consider in this situation the subalgebra of $\mathfrak{G}'$ of $\mathfrak{G}$ defined as

$$\mathfrak{G}' = \bigoplus_{k \in \mathbb{Z}} \mathfrak{G}_{kL} \tag{3}$$

and the corresponding subgroup $\mathcal{G}'$ of the group $\mathcal{G}$. The expansion (3) defines a $\mathbb{Z}$-gradation of $\mathfrak{G}'$ with the grading subspaces $\mathfrak{G}'_k = \mathfrak{G}_{kL}$, $k \in \mathbb{Z}$. It is evident that the Toda equation associated with the Lie group $\mathcal{G}$ and such choice of the positive integer L can be considered as the Toda equation associated with the Lie group $\mathcal{G}'$ and the choice $L = 1$.

If there is an isomorphism F from a $\mathbb{Z}$–graded Lie algebra $\mathfrak{G}$ to a $\mathbb{Z}$–graded Lie algebra $\mathfrak{H}$ relating the corresponding grading subspaces as $\mathfrak{H}_k = F(\mathfrak{G}_k)$, one says that $\mathbb{Z}$-gradations of $\mathfrak{G}$ and $\mathfrak{H}$ are conjugated by F. Actually, having a $\mathbb{Z}$-graded Lie algebra $\mathfrak{G}$, one can induce a $\mathbb{Z}$-gradation of an F-isomorphic Lie algebra $\mathfrak{H}$, using $\mathfrak{H}_k = F(\mathfrak{G}_k)$ as its grading subspaces.

It is clear that conjugated $\mathbb{Z}$-gradations give actually the same Toda equations. Therefore, to perform a classification of Toda equations associated with the Lie group $\mathcal{G}$ one should classify non-conjugated $\mathbb{Z}$-gradations of its Lie algebra $\mathfrak{G}$. In the case when $\mathfrak{G}$ is a complex classical Lie group a convenient classification of $\mathbb{Z}$-gradations was described and the corresponding Toda equations were presented in the paper [14], see also [12, 13]. Here we review and discuss the corresponding results obtained in the papers [16, 17] for the case when $\mathcal{G}$ is a loop group of a complex classical Lie group.

2.2 Loop Lie algebras and loop groups

Let $\mathfrak{g}$ be a finite dimensional real or complex Lie algebra. The *loop Lie algebra* of $\mathfrak{g}$, denoted $\mathcal{L}(\mathfrak{g})$, is usually defined as the linear space $C^\infty(S^1, \mathfrak{g})$ of smooth mappings from the circle S^1 to $\mathfrak{g}$ with the Lie algebra operation defined pointwise. In this paper we define $\mathcal{L}(\mathfrak{g})$ as the linear space $C^\infty_{2\pi}(\mathbb{R}, \mathfrak{g})$ of smooth 2π-periodic mappings of the real line $\mathbb{R}$ to $\mathfrak{g}$ with the Lie algebra operation again defined pointwise. One can show that these two Lie algebras are isomorphic. We assume that $\mathcal{L}(\mathfrak{g})$ is supplied with the structure of a Fréchet space[2] in such a way that the Lie algebra operation is continuous, see, for example, [16, 18, 19]. We call such a Lie algebra a *Fréchet Lie algebra*.

Now, let G be a Lie group with the Lie algebra $\mathfrak{g}$. The loop group of G, denoted $\mathcal{L}(G)$, is defined alternatively either as the set $C^\infty(S^1, G)$ of smooth mappings from S^1 to G or as the set $C^\infty_{2\pi}(\mathbb{R}, G)$ of smooth 2π-periodic mappings from $\mathbb{R}$ to G with the group law defined in both cases pointwise. In this paper we adopt the second definition. We assume that $\mathcal{L}(G)$ is supplied with the structure of a Fréchet manifold modeled on $\mathcal{L}(\mathfrak{g})$ in such a way that it becomes a Lie group, see, for example, [16, 18, 19]. Here the Lie algebra of the Lie group $\mathcal{L}(G)$ is naturally identified with the loop Lie algebra $\mathcal{L}(\mathfrak{g})$.

It is not difficult to generalize the above definitions to the case of twisted loop Lie algebras and loop groups. Let A be an automorphism of a Lie algebra $\mathfrak{g}$ satisfying the relation $A^M = \mathrm{id}_\mathfrak{g}$ for some positive integer M. The *twisted loop Lie algebra* $\mathcal{L}_{A,M}(\mathfrak{g})$ is

[2]A *Fréchet space* is a complete topological vector space, whose topology is induced by a countable collection of seminorms.

a subalgebra of the loop Lie algebra $\mathcal{L}(\mathfrak{g})$ formed by the elements ξ which satisfy the equality $\xi(\sigma + 2\pi/M) = A(\xi(\sigma))$. Similarly, given an automorphism a of a Lie group G which satisfies the relation $a^M = \mathrm{id}_G$, we define the *twisted loop group* $\mathcal{L}_{a,M}(G)$ as the subgroup of the loop group $\mathcal{L}(G)$ formed by the elements ρ satisfying the equality $\rho(\sigma + 2\pi/M) = a(\rho(\sigma))$. The Lie algebra of a twisted loop group $\mathcal{L}_{a,M}(G)$ is naturally identified with the twisted loop Lie algebra $\mathcal{L}_{A,M}(\mathfrak{g})$, where we denote the automorphism of the Lie algebra $\mathfrak{g}$ corresponding to the automorphism a of the Lie group G by A.

It is clear that a loop Lie algebra $\mathcal{L}(\mathfrak{g})$ can be treated as a twisted loop Lie algebra $\mathcal{L}_{\mathrm{id}_\mathfrak{g},M}(\mathfrak{g})$, where M is an arbitrary positive integer. In its turn, a loop group $\mathcal{L}(G)$ can be treated as a twisted loop group $\mathcal{L}_{\mathrm{id}_G,M}(G)$, where M is again an arbitrary positive integer. In the present paper by loop Lie algebras and loop groups we mean twisted loop Lie algebras and twisted loop groups.

2.3 $\mathbb{Z}$-gradations of loop Lie algebras and corresponding Toda equations

First, let us recall the method used in the paper [14] to describe $\mathbb{Z}$-gradations of the Lie algebras of the complex classical Lie groups. By these groups we mean the Lie groups $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{O}_n(\mathbb{C})$ and $\mathrm{Sp}_n(\mathbb{C})$, whose Lie algebras are $\mathfrak{gl}_n(\mathbb{C})$, $\mathfrak{so}_n(\mathbb{C})$ and $\mathfrak{sp}_n(\mathbb{C})$ respectively, see Section 3.1.

We start with a little wider class of Lie algebras. Let $\mathfrak{G}$ be a finite dimensional complex simple Lie algebra endowed with a $\mathbb{Z}$-gradation. Define a linear operator Q acting on an element $\xi \in \mathfrak{G}$ as

$$Q\xi = \sum_{k \in \mathbb{Z}} k\xi_k, \tag{4}$$

where ξ_k, $k \in \mathbb{Z}$, are the grading components of ξ. It is clear that

$$\mathfrak{G}_k = \{\xi \in \mathfrak{G} \mid Q\xi = k\xi\}.$$

Thus, the operator Q completely determines the corresponding $\mathbb{Z}$-gradation. It is called the grading operator generating the $\mathbb{Z}$-gradation under consideration.

One can easily show that Q is a derivation of the Lie algebra $\mathfrak{G}$. It is well known that any derivation of a complex simple Lie algebra is an inner derivation. Therefore, there is a unique element $q \in \mathfrak{G}$ such that $Q\xi = [q, \xi]$. Hence, the problem of classification of $\mathbb{Z}$-gradations of $\mathfrak{G}$ in the case under consideration is reduced to the problem of classification of the elements $q \in \mathfrak{G}$ such that the operator $\mathrm{ad}(q)$ is semisimple and has only integer eigenvalues. It is known that any such element belongs to some Cartan subalgebra of $\mathfrak{G}$. Since all Cartan subalgebras are conjugated by inner automorphisms of $\mathfrak{G}$, to classify the $\mathbb{Z}$-gradations of $\mathfrak{G}$ up to conjugations one can assume that the element q belongs to some fixed Cartan subalgebra.

If $\mathfrak{G}$ is a complex classical Lie algebra it is convenient to work with the Cartan subalgebra formed by diagonal matrices. After that the problem of classification of $\mathbb{Z}$-gradations of $\mathfrak{G}$ becomes almost trivial, and its results can be visually represented with the help of the corresponding block matrix decompositions of the elements of $\mathfrak{G}$ which appear to be very convenient for description of the Toda systems associated with complex classical Lie groups [14].

The main lesson here is that it is useful to describe $\mathbb{Z}$-gradations of a Lie algebra $\mathfrak{G}$ by their grading operators being special cases of derivations of $\mathfrak{G}$. In the case of infinite

dimensional Fréchet Lie algebras one should have in mind that the requirement of continuity is included into the definition of a derivation. When this requirement is rejected, it seems impossible to obtain substantial results on the form of derivations.

Let now $\mathcal{G}$ be an infinite dimensional Fréchet Lie group and $\mathfrak{G}$ be its Fréchet Lie algebra. For a general $\mathbb{Z}$-gradation of $\mathfrak{G}$ one cannot use the relation (4) to define a linear operator in $\mathfrak{G}$ because the series in the right hand side of the relation (4) may diverge for some ξ. We say that the $\mathbb{Z}$-gradation under consideration is generated by grading operator if this series converges absolutely for every $\xi \in \mathfrak{G}$.

In the case when $\mathfrak{G} = \mathcal{L}_{A,M}(\mathfrak{g})$ the basic example is the standard $\mathbb{Z}$-gradation generated by the grading operator $Q = -\mathrm{i}\,\mathrm{d}/\mathrm{d}s$. Here the grading subspaces are

$$\mathcal{L}_{A,M}(\mathfrak{g})_k = \{\xi \in \mathcal{L}_{A,M}(\mathfrak{g}) \mid \xi = \mathrm{e}^{\mathrm{i}ks} x,\ x \in \mathfrak{g},\ A(x) = \mathrm{e}^{2\pi \mathrm{i}k/M} x\}.$$

Let $\mathfrak{G}$ be supplied with a $\mathbb{Z}$-gradation which is generated by the grading operator Q. It can be shown that in this case

$$Q[\xi, \eta] = [Q\xi, \eta] + [\xi, Q\eta]$$

for any $\xi, \eta \in \mathfrak{G}$. Hence, if the grading operator is continuous it is a derivation of $\mathfrak{G}$.

Now we restrict our consideration to the case when $\mathfrak{G}$ is a loop Lie algebra $\mathcal{L}_{A,M}(\mathfrak{g})$ with $\mathfrak{g}$ being a finite dimensional complex simple Lie algebra. Assume that $\mathcal{L}_{A,M}(\mathfrak{g})$ is endowed with a $\mathbb{Z}$-gradation which is generated by the continuous grading operator Q which is in such case a derivation of $\mathcal{L}_{A,M}(\mathfrak{g})$. The derivations of $\mathcal{L}_{A,M}(\mathfrak{g})$ can be described explicitly [16], and this allows one to write[3]

$$Q\xi = -\mathrm{i}X(\xi) + \mathrm{i}[\eta, \xi], \tag{5}$$

where X is a smooth $2\pi/M$-periodic complex vector field on $\mathbb{R}$ and η is an element of $\mathcal{L}_{A,M}(\mathfrak{g})$. To go further, it is desirable to show that X is a real vector field. It can be done if we restrict ourselves to the case of the so-called integrable $\mathbb{Z}$-gradations [16].

We call a $\mathbb{Z}$-gradation of a Fréchet Lie algebra $\mathfrak{G}$ integrable if the mapping $\Phi\colon \mathbb{R} \times \mathfrak{G} \to \mathfrak{G}$ defined by the relation

$$\Phi(\tau, \xi) = \sum_{k \in \mathbb{Z}} \mathrm{e}^{-\mathrm{i}k\tau} \xi_k$$

is smooth. Here as usual we denote by ξ_k the grading components of the element ξ. Any such gradation is generated by a continuous grading operator Q acting on an element $\xi \in \mathfrak{G}$ as

$$Q\xi = \mathrm{i} \left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_0 \Phi_\xi. \tag{6}$$

Here t is a standard coordinate on $\mathbb{R}$ and the smooth mapping $\Phi_\xi\colon \mathbb{R} \to \mathfrak{G}$ is defined by the equality $\Phi_\xi(\tau) = \Phi(\tau, \xi)$. Furthermore, for any fixed $\tau \in \mathbb{R}$ the mapping $\Phi_\tau\colon \xi \in \mathfrak{G} \mapsto \Phi(\tau, \xi)$ is an automorphism of $\mathfrak{G}$, and all such automorphisms form a one-parameter subgroup of the group of automorphisms of $\mathfrak{G}$.

Return again to the case of a loop Lie algebra $\mathcal{L}_{A,M}(\mathfrak{g})$ with $\mathfrak{g}$ being a finite dimensional complex simple Lie algebra and assume that it is supplied with an integrable

[3]The signs at the right hand side of the relation (5) are chosen for future convenience. The same reason explains explicit appearance of the imaginary unit.

$\mathbb{Z}$-gradation. Using the explicit form of the automorphisms of $\mathcal{L}_{A,M}(\mathfrak{g})$ [16] and the equality (6) one sees that in this case the grading operator Q is defined by the relation (5) where the vector field X is real. Here one can show that the vector field X is either a zero vector field or it has no zeros [16]. In the case when X is a zero vector field some of the grading subspaces are infinite dimensional. Let us restrict ourselves to the case of $\mathbb{Z}$-gradations with finite dimensional grading subspaces. In this case one can assume that the function $X(s)$, where s is a standard coordinate on $\mathbb{R}$, is positive. Indeed, in the case when $X(s)$ is negative one can conjugate the considered $\mathbb{Z}$-gradation by the isomorphism sending an element $\xi \in \mathcal{L}_{A,M}(\mathfrak{g})$ into the element $\xi' \in \mathcal{L}_{A^{-1},M}(\mathfrak{g})$ such that $\xi'(\sigma) = \xi(-\sigma)$. It is clear that this transformation inverses the sign of the vector field X.

Now, using conjugations by isomorphisms, we try to make the grading operator Q given by the relation (5) as simple as possible. To this end consider a mapping F from $\mathcal{L}_{A,M}(\mathfrak{g})$ into $C^\infty(\mathbb{R}, \mathfrak{g})$ defined by the equality

$$F\xi = \rho(f^{-1*}\xi)\rho^{-1},$$

where f is a diffeomorphism of $\mathbb{R}$, and ρ is an element of $C^\infty(\mathbb{R}, G)$. The mapping F is injective and can be considered an isomorphism from $\mathcal{L}_{A,M}(\mathfrak{g})$ to $F(\mathcal{L}_{A,M}(\mathfrak{g}))$. One can show that

$$FQF^{-1}\xi = -\mathrm{i}X'(\xi) + \mathrm{i}[\rho\eta'\rho^{-1} + X'(\rho)\rho^{-1}, \xi],$$

where $X' = f_*X$ and $\eta' = f^{-1*}\eta$.

Choose the diffeomorphism f so that $f_*X = \mathrm{d}/\mathrm{d}s$. To this end it suffices to define it by the relation

$$f(\sigma) = \int_{(0,\sigma)} \mathrm{d}s/X(s).$$

It is important here that the vector field X has no zeros. Since the vector field X is $2\pi/M$-periodic, one has

$$f(\sigma + 2\pi/M) = f(\sigma) + 2\pi/M',$$

where M' is a positive real integer such that

$$2\pi/M' = \int_{(0,2\pi/M)} \mathrm{d}s/X(s).$$

This equality implies, in particular, that

$$\eta'(\sigma + 2\pi/M') = A(\eta'(\sigma)). \tag{7}$$

Assume now that the mapping ρ is a solution of the equation

$$\rho^{-1}\mathrm{d}\rho/\mathrm{d}s = -\eta'. \tag{8}$$

It is well known that this equation always has solutions, all its solutions are smooth, and if ρ and ρ' are two solutions then $\rho' = g\rho$ for some $g \in G$. From the relation (7) it follows that if ρ is a solution of (8) then the mapping ρ' defined by the equality $\rho'(\sigma) = a^{-1}(\rho(\sigma + 2\pi/M'))$ is also a solution of (8). Therefore, for some $g \in G$ one has

$$\rho(\sigma + 2\pi/M') = a(g\rho(\sigma)).$$

It is not difficult to get convinced that with the choice of f and ρ described above the mapping F maps $\mathcal{L}_{A,M}(\mathfrak{g})$ isomorphically onto the Fréchet Lie algebra $\mathfrak{G}$ formed by smooth mappings ξ from $\mathbb{R}$ to $\mathfrak{g}$ satisfying the condition

$$\xi(\sigma + 2\pi/M') = A'(\xi(\sigma)),$$

where the automorphism A' is defined as $A' = A \circ \mathrm{Ad}(g)$. The grading operator FQF^{-1} generating the conjugated gradation of $\mathfrak{G}$ is just $-\mathrm{id}/\mathrm{d}s$. It is not difficult to show [16] that M' is an integer and that $A'^{M'} = \mathrm{id}_{\mathfrak{g}}$. It means that $\mathfrak{G} = \mathcal{L}_{A',M'}(\mathfrak{g})$.

Thus, we see that any integrable $\mathbb{Z}$-gradation of the loop Lie algebra $\mathcal{L}_{A,M}(\mathfrak{g})$ with finite dimensional grading subspaces is conjugated by an isomorphism to the standard gradation of another twisted loop Lie algebra $\mathcal{L}_{A',M'}(\mathfrak{g})$, where the automorphisms A and A' differ by an inner automorphism of $\mathfrak{g}$. Recall that we consider the case when $\mathfrak{g}$ is a finite dimensional complex simple Lie algebra.

Let G be a finite dimensional complex simple Lie group, a be an automorphism of G of order M, and A be the corresponding automorphism of the Lie algebra $\mathfrak{g}$ of G. To classify Toda equations associated with $\mathcal{L}_{a,M}(G)$ one should classify, up to conjugation by isomorphisms, $\mathbb{Z}$-gradations of $\mathcal{L}_{A,M}(\mathfrak{g})$. As was actually shown above, if we restrict ourselves to integrable $\mathbb{Z}_M$-gradations with finite dimensional grading subspaces this task is equivalent to classification of the loop groups $\mathcal{L}_{a,M}(G)$ themselves, or, equivalently, to classification of finite order automorphisms of G. It is not difficult to get convinced [17] that it suffices to perform the latter classification also up to conjugation by isomorphisms. As a matter of fact, we will classify finite order automorphisms of the Lie algebra $\mathfrak{g}$ which can be lifted to automorphisms of the Lie group G.

It is very useful to realize that every automorphism A of $\mathfrak{g}$ satisfying the relation $A^M = \mathrm{id}_{\mathfrak{g}}$ induces a $\mathbb{Z}_M$-gradation of $\mathfrak{g}$ with the grading subspaces[4]

$$\mathfrak{g}_{[k]_M} = \{x \in \mathfrak{g} \mid A(x) = \mathrm{e}^{2\pi \mathrm{i} k/M} x\}, \qquad k = 1, \ldots, M-1.$$

Vice versa, any $\mathbb{Z}_M$-gradation of $\mathfrak{g}$ defines in an evident way an automorphism A of $\mathfrak{g}$ satisfying the relation $A^M = \mathrm{id}_{\mathfrak{g}}$. A $\mathbb{Z}_M$-gradation of $\mathfrak{g}$ is called an inner or outer type gradation, if the associated automorphism A of $\mathfrak{g}$ is of inner or outer type respectively.

The grading subspaces of the standard $\mathbb{Z}$-gradation of the twisted loop Lie algebra $\mathcal{L}_{A,M}(\mathfrak{g})$ can be described in terms of the corresponding $\mathbb{Z}_M$-gradation of $\mathfrak{g}$ as

$$\mathcal{L}_{A,M}(\mathfrak{g})_k = \{\xi \in \mathcal{L}_{A,M}(\mathfrak{g}) \mid \xi = \mathrm{e}^{\mathrm{i}ks} x,\ x \in \mathfrak{g}_{[k]_M}\}.$$

For the standard gradation the positive integer L entering the definition of a Toda equation satisfies the inequality $L \leq M$, and the equality $L = M$ takes place if and only if $A = \mathrm{id}_{\mathfrak{g}}$, with the positive integer M being arbitrary. In this case the nontrivial grading subspaces are $\mathcal{L}_{\mathrm{id}_{\mathfrak{g}},M}(\mathfrak{g})_{kM}$ for $k \in \mathbb{Z}$, and one has

$$\mathcal{L}_{\mathrm{id}_{\mathfrak{g}},M}(\mathfrak{g})_{kM} = \{\xi \in \mathcal{L}_{\mathrm{id}_{\mathfrak{g}},M}(\mathfrak{g}) \mid \xi = \mathrm{e}^{\mathrm{i}kMs} x,\ x \in \mathfrak{g}\}.$$

It is also clear that for the standard $\mathbb{Z}$-gradation the subalgebra $\mathcal{L}_{A,M}(\mathfrak{g})_0$ is isomorphic to $\mathfrak{g}_{[0]_M}$, and the Lie group $\mathcal{L}_{a,M}(G)_0$ is isomorphic to the connected Lie subgroup G_0 of G corresponding to the Lie algebra $\mathfrak{g}_{[0]_M}$. Hence, the mapping Ξ is actually a

[4] We denote by $[k]_M$ the element of the ring $\mathbb{Z}_M$ corresponding to the integer k.

mapping from $\mathcal{M}$ to G_0, for consistency with the notation used earlier we will denote it by γ. The mappings $\mathcal{F}_-$ and $\mathcal{F}_+$ are given by the relation

$$\mathcal{F}_-(p) = \mathrm{e}^{-\mathrm{i}Ls} c_-(p), \qquad \mathcal{F}_+(p) = \mathrm{e}^{\mathrm{i}Ls} c_+(p), \qquad p \in \mathcal{M},$$

where c_- and c_+ are mappings from $\mathcal{M}$ to $\mathfrak{g}_{-[L]_M}$ and $\mathfrak{g}_{+[L]_M}$ respectively. Thus, the Toda equation (1) can be written as

$$\partial_+(\gamma^{-1}\partial_-\gamma) = [c_-, \gamma^{-1}c_+\gamma], \tag{9}$$

where γ is a smooth mapping from $\mathcal{M}$ to G_0, and the mappings c_- and c_+ are fixed smooth mappings from $\mathcal{M}$ to $\mathfrak{g}_{-[L]_M}$ and $\mathfrak{g}_{+[L]_M}$ respectively. The conditions (2) imply that

$$\partial_+ c_- = 0, \qquad \partial_- c_+ = 0. \tag{10}$$

Summarizing one can say that a Toda equation associated with a loop group of a simple complex Lie group whose Lie algebra is endowed with an integrable $\mathbb{Z}$-gradation with finite dimensional grading subspaces is equivalent to the equation of the form (9).

It is reasonable to single out the simplest case, with A being $\mathrm{id}_\mathfrak{g}$ and M an arbitrary positive number. In this case $L = M$. The mapping γ is a mapping from $\mathcal{M}$ to the whole group G, and c_+ and c_- are mappings from $\mathcal{M}$ to $\mathfrak{g}$. Denoting in this particular case γ by $\mathit{\Gamma}$, c_+ by C_+, and c_- by C_-, one writes the Toda equation (9) as

$$\partial_+(\mathit{\Gamma}^{-1}\partial_-\mathit{\Gamma}) = [C_-, \mathit{\Gamma}^{-1}C_+\mathit{\Gamma}], \tag{11}$$

and the conditions (10) as

$$\partial_+ C_- = 0, \qquad \partial_- C_+ = 0. \tag{12}$$

Note that this example indicates a principal difference between Toda systems associated with finite dimensional and loop Lie groups.

One can consider equations of the type (9) in a more general setting. Namely, let G be an arbitrary finite dimensional Lie group and a be an arbitrary finite order automorphism of G. The corresponding automorphism A of the Lie algebra $\mathfrak{g}$ of the Lie group G generates a $\mathbb{Z}_M$-gradation of $\mathfrak{g}$. Assume that for some positive integer $L \le M$ the grading subspaces $\mathfrak{g}_{+[k]_M}$ and $\mathfrak{g}_{-[k]_M}$ for $0 < k < L$ are trivial. Choose some fixed mappings c_+ and c_- from $\mathcal{M}$ to $\mathfrak{g}_{+[L]_M}$ and $\mathfrak{g}_{-[L]_M}$, respectively, satisfying the relations (10). Now the equation (9) is equivalent to a Toda equation associated with the loop group $\mathcal{L}_{a,M}(G)$ whose Lie algebra $\mathcal{L}_{A,M}(\mathfrak{g})$ is endowed with the standard $\mathbb{Z}$-gradation.

Note that the authors of the paper [11] also suggest the equation (9) as a convenient form of a Toda equation associated with a loop group. They do not assume that the grading subspaces $\mathfrak{g}_{+[k]_M}$ and $\mathfrak{g}_{-[k]_M}$ for $0 < k < L$ are trivial. Actually their assumption does not lead to new Toda equations.

Indeed, suppose that we do not assume that the grading subspaces $\mathfrak{g}_{+[k]_M}$ and $\mathfrak{g}_{-[k]_M}$ for $0 < k < L$ are trivial. As in Section 2.1 consider a subalgebra

$$\mathfrak{G}' = \bigoplus_{k \in Z} \mathcal{L}_{A,M}(\mathfrak{g})_{kL}$$

of the loop Lie algebra $\mathcal{L}_{A,M}(\mathfrak{g})$ as a $\mathbb{Z}$-graded Lie algebra with the grading subspaces $\mathfrak{G}'_k = \mathcal{L}_{A,M}(\mathfrak{g})_{kL}$. The Toda equation associated with the Lie group $\mathcal{L}_{a,M}(G)$ and the

above choice of the positive integer L can be considered as the Toda equation associated with the Lie group $\mathcal{G}'$ corresponding to the Lie algebra $\mathfrak{G}'$ and the choice $L = 1$. Let us show that the Lie algebra $\mathfrak{G}'$ is isomorphic to some loop Lie algebra.

Let M' be the minimal positive integer such that $M'[L]_M = [0]_M$. In other words, M' is the order of the element $[L]_M$ considered as an element of the additive group of the ring $\mathbb{Z}_M$. Consider the subalgebra $\mathfrak{g}'$ of the Lie algebra $\mathfrak{g}$ defined as

$$\mathfrak{g}' = \bigoplus_{k=0}^{M'-1} \mathfrak{g}_{k[L]_M}.$$

The Lie algebra $\mathfrak{g}'$ can be treated as a $\mathbb{Z}_{M'}$-graded Lie algebra with the grading subspaces $\mathfrak{g}'_{[k]_{M'}} = \mathfrak{g}_{k[L]_M}$. Denote the corresponding automorphism of $\mathfrak{g}'$ by A'.

Every element $\xi \in \mathfrak{G}'$ can be represented as the absolutely convergent sum

$$\xi = \sum_{k \in \mathbb{Z}} \mathrm{e}^{\mathrm{i}kLs} x_k,$$

where for each $k \in \mathbb{Z}$ one has $x_k \in \mathfrak{g}_{k[L]_M}$. It is clear that the series

$$\xi' = \sum_{k \in \mathbb{Z}} \mathrm{e}^{\mathrm{i}ks} x_k$$

is absolutely convergent, and ξ' can be considered as an element of $\mathcal{L}_{A',M'}(\mathfrak{g})$. It can be easily verified that the mapping sending ξ to ξ' is an isomorphism from $\mathfrak{G}'$ to $\mathcal{L}_{A',M'}(\mathfrak{g})$.

Thus, if we do not not assume that the grading subspaces $\mathfrak{g}_{+[k]_M}$ and $\mathfrak{g}_{-[k]_M}$ for $0 < k < L$ are trivial, then the arising Toda equation associated with the loop group $\mathcal{L}_{A,M}(G)$ can be considered as the Toda equation associated with some other loop group $\mathcal{L}_{A',M'}(G')$ for the case when $L = 1$. Here G' is the Lie group corresponding to the subalgebra $\mathfrak{g}'$ of the Lie algebra $\mathfrak{g}$.

In what follows, explicit forms which the Toda equation (9) takes for complex classical Lie groups G are specified. It was pointed out that this specification should use the classification, up to conjugations, of the finite order automorphisms of the Lie algebras under consideration. Instead of using root techniques as, for example, in [20–22], here the classification in terms of convenient block matrix representations is implemented.

3 $\mathbb{Z}_M$-gradations of complex classical Lie algebras

3.1 Complex classical Lie groups and Lie algebras

Here we define the complex classical Lie groups and discuss their basic properties. More information on these groups can be found, for example, in the works [17, 20, 22].

Let us first explain the notation used. We denote by I_n the unit diagonal $n \times n$ matrix and by J_n the symmetric skew diagonal $n \times n$ matrix. For an even n we also define the skew symmetric skew diagonal $n \times n$ matrix

$$K_n = \begin{pmatrix} 0 & J_{n/2} \\ -J_{n/2} & 0 \end{pmatrix}.$$

When it does not lead to a misunderstanding, we write instead of I_n, J_n, and K_n just I, J, and K respectively.

Besides, the following convention is being used. If m and B be $n \times n$ matrices, one denotes ${}^B m = B^{-1}\,{}^t m B$, where ${}^t m$ is the transpose of the matrix m. Note that ${}^J m$ is actually the transpose of m with respect to the skew diagonal.

The complex general linear group $\mathrm{GL}_n(\mathbb{C})$ is formed by all nonsingular complex $n \times n$ matrices with matrix multiplication as the group law. The Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ of the Lie group $\mathrm{GL}_n(\mathbb{C})$ is formed by all complex $n \times n$ matrices with matrix commutator as the Lie algebra law. The Lie group $\mathrm{GL}_n(\mathbb{C})$ and the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ are not simple. The subgroup $\mathrm{SL}_n(\mathbb{C})$ of $\mathrm{GL}_n(\mathbb{C})$ formed by the matrices with unit determinant is called the complex special linear group. The Lie group $\mathrm{SL}_n(\mathbb{C})$ is connected and simple. Its Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ is also simple.

Let B be a complex nonsingular $n \times n$ matrix. The elements g of $\mathrm{GL}_n(\mathbb{C})$ singled out by the condition ${}^B g = g^{-1}$ form a Lie subgroup of $\mathrm{GL}_n(\mathbb{C})$ which one denotes by $\mathrm{GL}_n^B(\mathbb{C})$. The Lie algebra $\mathfrak{gl}_n^B(\mathbb{C})$ of $\mathrm{GL}_n^B(\mathbb{C})$ is a subalgebra of the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ formed by the complex $n \times n$ matrices x satisfying the condition ${}^B x = -x$.

For any symmetric nonsingular $n \times n$ matrix B the Lie group $\mathrm{GL}_n^B(\mathbb{C})$ is isomorphic to the Lie group $\mathrm{GL}_n^J(\mathbb{C})$. This group is called the complex orthogonal group and is denoted $\mathrm{O}_n(\mathbb{C})$. For an element $g \in \mathrm{O}_n(\mathbb{C})$ from the equality ${}^J g = g^{-1}$ one obtains that $\det g$ is equal either to 1 or to -1. The elements of $\mathrm{O}_n(\mathbb{C})$ with unit determinant form a connected Lie subgroup of $\mathrm{O}_n(\mathbb{C})$ called the complex special orthogonal group and is denoted $\mathrm{SO}_n(\mathbb{C})$. This subgroup is the connected component of the identity of $\mathrm{O}_n(\mathbb{C})$. The Lie algebra of $\mathrm{SO}_n(\mathbb{C})$ is denoted $\mathfrak{so}_n(\mathbb{C})$. It is clear that the Lie algebra of $\mathrm{O}_n(\mathbb{C})$ coincides with the Lie algebra of $\mathrm{SO}_n(\mathbb{C})$. The Lie group $\mathrm{SO}_n(\mathbb{C})$ and the Lie algebra $\mathfrak{so}_n(\mathbb{C})$ are simple. The special orthogonal Lie group and its Lie algebra possess both inner and outer automorphisms, the latter exist only when n is even.

For an even n take a skew symmetric nonsingular $n \times n$ matrix B. In this case the Lie group $\mathrm{GL}_n^B(\mathbb{C})$ is isomorphic to the Lie group $\mathrm{GL}_n^K(\mathbb{C})$. This group is called the complex symplectic group and is denoted $\mathrm{Sp}_n(\mathbb{C})$. The Lie group $\mathrm{Sp}_n(\mathbb{C})$ is connected and simple. The corresponding Lie algebra $\mathfrak{sp}_n(\mathbb{C})$ is also simple. Remember that the symplectic Lie group and its Lie algebra have only inner automorphisms.

3.2 $\mathbb{Z}_M$-gradations of complex general linear Lie algebras of inner type

In this section we consider inner type $\mathbb{Z}_M$-gradations of complex general linear Lie algebras. We call such gradations gradations of $\mathfrak{gl}_n(\mathbb{C})$ of type I. There are two more types of $\mathbb{Z}_M$-gradations generated by outer automorphisms of complex general linear Lie algebras. They will be considered in Section 3.4.

Type I

Let a be an inner automorphism of the Lie group $\mathrm{GL}_n(\mathbb{C})$ satisfying the relation $a^M = \mathrm{id}_{\mathrm{GL}_n(\mathbb{C})}$. Denote the corresponding inner automorphism of the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ by A. This automorphism satisfies the relation $A^M = \mathrm{id}_{\mathfrak{gl}_n(\mathbb{C})}$. In other words, A is a finite order automorphism of $\mathfrak{gl}_n(\mathbb{C})$. Since one is interested in the automorphisms of $\mathfrak{gl}_n(\mathbb{C})$ up to conjugations, it can be assumed that the automorphism A under consideration

is given by the relation

$$A(x) = hxh^{-1}, \tag{13}$$

where h is an element of the subgroup $\mathrm{D}_n(\mathbb{C})$ of $\mathrm{GL}_n(\mathbb{C})$ formed by all complex non-singular diagonal matrices, see, for example, [17]. It is clear that multiplying h by an arbitrary nonzero complex number one obtains an element of $\mathrm{D}_n(\mathbb{C})$ which generates the same automorphism of $\mathfrak{gl}_n(\mathbb{C})$ as the initial element.

The equality $A^M = \mathrm{id}_{\mathfrak{gl}_n(\mathbb{C})}$ gives $h^M x h^{-M} = x$ for any $x \in \mathfrak{gl}_n(\mathbb{C})$. Therefore, $h^M = \nu I$, where ν is a nonzero complex number. Multiplying h by an appropriate complex number we make it satisfy the relation $h^M = I$. This means that the diagonal matrix elements of h have the form $\mathrm{e}^{2\pi \mathrm{i} m/M}$, where m is an integer. We will assume that $0 < m \leq M$. Using inner automorphisms of $\mathfrak{gl}_n(\mathbb{C})$ which permute the rows and columns of the matrix h synchronously, we collect coinciding diagonal matrix elements together, and come to the following block diagonal form of the element h:

$$h = \begin{pmatrix} \mu_1 I_{n_1} & & & \\ & \mu_2 I_{n_2} & & \\ & & \ddots & \\ & & & \mu_p I_{n_p} \end{pmatrix}. \tag{14}$$

Here $\mu_\alpha = \mathrm{e}^{2\pi \mathrm{i} m_\alpha/M}$, the positive integers m_α form a decreasing sequence, $M \geq m_1 > m_2 > \ldots > m_p > 0$, and the positive integers n_α satisfy the equality $\sum_\alpha^p n_\alpha = n$. It is assumed that the integer p is greater than 1. The case $p = 1$ corresponds to $A = \mathrm{id}_{\mathfrak{gl}_n(\mathbb{C})}$, and one reveals the equation (11), where Γ is a mapping from $\mathcal{M}$ to $\mathrm{GL}_n(\mathbb{C})$, and C_- and C_+ are mappings from $\mathcal{M}$ to $\mathfrak{gl}_n(\mathbb{C})$ satisfying the conditions (12).

Now consider the corresponding $\mathbb{Z}$-gradation. Represent the general element x of $\mathfrak{gl}_n(\mathbb{C})$ in the block matrix form suggested by the structure of h,

$$x = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{p1} & x_{p2} & \cdots & x_{pp} \end{pmatrix}, \tag{15}$$

where $x_{\alpha\beta}$, $\alpha, \beta = 1, \ldots, p$, is an $n_\alpha \times n_\beta$ matrix. One easily finds

$$(hxh^{-1})_{\alpha\beta} = \mathrm{e}^{2\pi \mathrm{i}(m_\alpha - m_\beta)/M} x_{\alpha\beta}.$$

Hence, if for fixed α and β only the block $x_{\alpha\beta}$ of the element x is different from zero, then x belongs to the grading subspace $[m_\alpha - m_\beta]_M$. It is convenient to introduce integers k_α, $\alpha = 1, \ldots, p-1$, defined as $k_\alpha = m_\alpha - m_{\alpha+1}$. By definition, for each α the integer k_α is positive and $\sum_{\alpha=1}^{p-1} k_\alpha = m_1 - m_p < M$. It is clear that for $\alpha < \beta$ one has $[m_\alpha - m_\beta]_M = [\sum_{\gamma=\alpha}^{\beta-1} k_\gamma]_M$, and for $\alpha > \beta$ one has $[m_\alpha - m_\beta]_M = -[\sum_{\gamma=\beta}^{\alpha-1} k_\gamma]_M = [M - \sum_{\gamma=\beta}^{\alpha-1} k_\gamma]_M$. These relations allow one to describe the grading structure of the $\mathbb{Z}_M$-gradation generated by the automorphism A by Figure 1. Here the elements of the ring $\mathbb{Z}_M$ are the grading indices of the corresponding blocks in the block matrix

$$\left(\begin{array}{c|c|c|c|c}
[0]_M & [k_1]_M & [k_1+k_2]_M & \cdots & [\sum\limits_{\alpha=1}^{p-1} k_\alpha]_M \\ \hline
-[k_1]_M & [0]_M & [k_2]_M & \cdots & [\sum\limits_{\alpha=2}^{p-1} k_\alpha]_M \\ \hline
-[k_1+k_2]_M & -[k_2]_M & [0]_M & \cdots & [\sum\limits_{\alpha=3}^{p-1} k_\alpha]_M \\ \hline
\vdots & \vdots & \vdots & \ddots & \vdots \\ \hline
-[\sum\limits_{\alpha=1}^{p-1} k_\alpha]_M & -[\sum\limits_{\alpha=2}^{p-1} k_\alpha]_M & -[\sum\limits_{\alpha=3}^{p-1} k_\alpha]_M & \cdots & [0]_M
\end{array}\right)$$

Figure 1: The canonical structure of a $\mathbb{Z}_M$-gradation

representation (15) of a general element of $\mathfrak{gl}_n(\mathbb{C})$. Note, in particular, that the subalgebra $\mathfrak{g}_{[0]_M}$ is formed by all block diagonal matrices and is isomorphic to the Lie algebra $\mathfrak{gl}_{n_1}(\mathbb{C}) \times \cdots \times \mathfrak{gl}_{n_p}(\mathbb{C})$. The group G_0 is also formed by block diagonal matrices, and as such, it is isomorphic to $\mathrm{GL}_{n_1}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{n_p}(\mathbb{C})$. This convenient structure of $\mathfrak{g}_{[0]_M}$ and G_0 is due to the chosen ordering of the diagonal matrix elements of h. Here one can say that the numbers μ_α are ordered clock wise as points on the unit circle in the complex plane.

It is clear that up to conjugations a $\mathbb{Z}_M$-gradation of $\mathfrak{gl}_n(\mathbb{C})$ of inner type can be specified by a choice of $p \leq n$ positive integers n_α, satisfying the equality $\sum_{\alpha=1}^{p} n_\alpha = n$, and $p-1$ positive integers k_α, satisfying the inequality $\sum_{\alpha=1}^{p-1} k_\alpha < M$. The corresponding automorphism of $\mathfrak{gl}_n(\mathbb{C})$ is defined by the relation (13), where h is given by the equality (14). Here $\mu_\alpha = \mathrm{e}^{2\pi \mathrm{i} m_\alpha / M}$ with

$$m_\alpha = \sum_{\beta=\alpha}^{p-1} k_\beta + m_p, \qquad \alpha = 1, \ldots, p-1, \tag{16}$$

while m_p is an arbitrary positive integer such that the inequality $\sum_{\alpha=1}^{p-1} k_\alpha < M$ is valid. Different choices of m_p give the same automorphism of $\mathfrak{gl}_n(\mathbb{C})$.

3.3 $\mathbb{Z}_M$-gradations of complex orthogonal and symplectic Lie algebras

Up to conjugations, all inner and outer type $\mathbb{Z}_M$-gradations of complex orthogonal and symplectic Lie algebras are generated by automorphisms given by the relation (13),[5] where h belongs either to the Lie group $\mathrm{O}_n(\mathbb{C}) \cap \mathrm{D}_n(\mathbb{C})$ or to the Lie group $\mathrm{Sp}_n(\mathbb{C}) \cap \mathrm{D}_n(\mathbb{C})$. Actually $\mathrm{O}_n(\mathbb{C}) \cap \mathrm{D}_n(\mathbb{C}) = \mathrm{Sp}_n(\mathbb{C}) \cap \mathrm{D}_n(\mathbb{C})$ and it is convenient to assume that $h \in \mathrm{O}_n(\mathbb{C}) \cap \mathrm{D}_n(\mathbb{C})$. For any element $x \in \mathfrak{gl}_n^B(\mathbb{C})$ one has $h^M x h^{-M} = x$, therefore, $h^M = \nu I$ for some complex number ν. This equality implies that $({}^B h)^M = \nu I$,

[5] Strictly speaking, for the Lie algebra $\mathfrak{so}_8(\mathbb{C})$ there are outer automorphisms which are not described by the relation (13). However, these automorphisms cannot be lifted up to automorphisms of the Lie group $\mathrm{SO}_8(\mathbb{C})$ and are not relevant for our purposes.

therefore, $({}^B h)^M h^M = \nu^2 I$. From the other hand, using the equality ${}^B h h = I$, one obtains $({}^B h)^M h^M = ({}^B h h)^M = I$. Thus, one sees that $\nu^2 = 1$. In other words, one has that ν is equal either to 1 or to -1. In both cases the diagonal matrix elements of h are of modulus one.

To come to the canonical structure of a $\mathbb{Z}_M$-gradation one has to bring h to the form (14) where the numbers μ_α are ordered clock wise as points on the unit circle in the complex plane. It appears that in some cases this cannot be done by automorphisms of the Lie algebra under consideration. Actually in these cases some diagonal matrix elements are equal to 1 and some of them are equal to -1 and one cannot collect such elements together keeping h in $\mathfrak{so}_n(\mathbb{C})$ or $\mathfrak{sp}_n(\mathbb{C})$. It may seem that similar obstructions for the required ordering arise also in the case when some diagonal matrix elements are equal to 1 even if there are no matrix elements equal to -1. However, in this case one can multiply h by -1 thus overcoming the problem. As a result we obtain two types of $\mathbb{Z}_M$-gradations of complex orthogonal and symplectic Lie algebras.

Type I

We start with the case when it is possible to perform the desired ordering of the numbers μ_α staying within $\mathfrak{so}_n(\mathbb{C})$ or $\mathfrak{sp}_n(\mathbb{C})$. As for the case of $\mathfrak{gl}_n(\mathbb{C})$ a $\mathbb{Z}_M$-gradation of $\mathfrak{so}_n(\mathbb{C})$ or $\mathfrak{sp}_n(\mathbb{C})$ can be specified by a choice of $p \le n$ positive integers n_α, satisfying the equality $\sum_{\alpha=1}^{p} n_\alpha = n$, and $p-1$ positive integers k_α, satisfying the inequality $\sum_{\alpha=1}^{p-1} k_\alpha < M$. But obviously these integers are no more as arbitrary as in the general linear case. They are subject to additional constraints coming from the corresponding Lie group and algebra defining conditions. Explicitly, the numbers n_α satisfy the equalities

$$n_{p-\alpha+1} = n_\alpha, \qquad \alpha = 1, \ldots, p,$$

and for the numbers k_α one has

$$k_{p-\alpha} = k_\alpha, \qquad \alpha = 1, \ldots, p-1.$$

Let $M - \sum_{\alpha=1}^{p-1} k_\alpha$ be an even positive integer. In this case the automorphism of $\mathfrak{so}_n(\mathbb{C})$ or $\mathfrak{sp}_n(\mathbb{C})$ generating the $\mathbb{Z}_M$-gradation under consideration is defined by the relation (13) with h given by the equality (14) where $\mu_\alpha = \mathrm{e}^{2\pi \mathrm{i} m_\alpha / M}$. The integer m_p is now fixed by the equality

$$m_p = \frac{1}{2}\left(M - \sum_{\alpha=1}^{p-1} k_\alpha\right),$$

and the integers m_α, $\alpha = 1, \ldots, p-1$, are given by the relation (16).

Let now $M - \sum_{\alpha=1}^{p-1} k_\alpha + 1$ be an even positive integer. In this case the automorphism of $\mathfrak{so}_n(\mathbb{C})$ or $\mathfrak{sp}_n(\mathbb{C})$ generating the $\mathbb{Z}_M$-gradation under consideration is defined by the relation (13) with h given by the equality (14) where $\mu_\alpha = \mathrm{e}^{2\pi \mathrm{i}(m_\alpha + 1/2)/M}$. The integer m_p is fixed by the equality

$$m_p = \frac{1}{2}\left(M - \sum_{\alpha=1}^{p-1} k_\alpha + 1\right),$$

and the integers m_α, $\alpha = 1, \ldots, p-1$, are again given by the relation (16).

In both cases the structure of the $\mathbb{Z}_M$-gradation under consideration is depicted by Figure 1. However, now one is faced with the necessity to impose the appropriate restrictions on the blocks of the decomposition (15). This implies in particular that the Lie algebra $\mathfrak{g}_{[0]_M}$ is formed by block diagonal matrices. This Lie algebra is isomorphic to $\mathfrak{gl}_{n_1}(\mathbb{C}) \times \cdots \times \mathfrak{gl}_{n_s}(\mathbb{C})$ for an even $p = 2s$, and it is isomorphic to $\mathfrak{gl}_{n_1}(\mathbb{C}) \times \cdots \times \mathfrak{gl}_{n_{s-1}}(\mathbb{C}) \times \mathfrak{gl}^{B_s}_{n_s}(\mathbb{C})$ for an odd $p = 2s - 1$. Here either $B_s = J_{n_s}$ or $B_s = K_{n_s}$ depending on what kind of Lie algebras is considered. It is clear that in the symplectic case for an odd p the integer n_s should be even. The Lie group G_0 is isomorphic either to $\mathrm{GL}_{n_1}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{n_s}(\mathbb{C})$ for an even $p = 2s$, or to $\mathrm{GL}_{n_1}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{n_{s-1}}(\mathbb{C}) \times \mathrm{GL}^{B_s}_{n_s}(\mathbb{C})$ for an odd $p = 2s - 1$.

Type II

This type corresponds to the case when some diagonal matrix elements of the element h generating the $\mathbb{Z}_M$-gradation under consideration are equal to 1 and some of them are equal to -1. Note that this is possible only if M even and also p is even.

Permuting the rows and columns of the matrices representing the elements of the Lie algebra under consideration in an appropriate way we move the diagonal matrix elements of h equal to 1 to the beginning of the diagonal. This transformation can be performed in such a way that the Lie algebra is mapped isomorphically onto the Lie algebra $\mathfrak{gl}^B_n(\mathbb{C})$ with

$$B = \begin{pmatrix} J_{n_1} & 0 \\ 0 & J_{n-n_1} \end{pmatrix}$$

in the orthogonal case, and

$$B = \begin{pmatrix} K_{n_1} & 0 \\ 0 & K_{n-n_1} \end{pmatrix}$$

in the symplectic case. Here n_1 is the number of diagonal matrix elements of h equal to 1. Then by an automorphism of $\mathfrak{gl}^B_n(\mathbb{C})$ we order the remaining diagonal matrix elements of h clock wise on the unit circle in the complex plane. As the result of our transformation we come to $\mathbb{Z}_M$-gradations which are specified by a set of $p \leq n$ positive integers n_α such that $\sum_{\alpha=1}^{p} n_\alpha = n$ and

$$n_{p-\alpha+2} = n_\alpha, \qquad \alpha = 2, \ldots, p,$$

and by a set of $p - 1$ positive integers k_α such that

$$\sum_{\alpha=1}^{p-1} k_\alpha + k_1 = M$$

and

$$k_{p-\alpha+1} = k_\alpha, \qquad \alpha = 2, \ldots, p - 1.$$

The automorphism generating the $\mathbb{Z}_M$-gradation under consideration is defined by the relation (13) with h given by the equality (14) where $\mu_\alpha = \mathrm{e}^{2\pi \mathrm{i} m_\alpha / M}$. Here $m_p = k_1$ and the integers m_α, $\alpha = 1, \ldots, p-1$, are given by the relation (16).

The structure of the $\mathbb{Z}_M$-gradation under consideration is depicted by Figure 1. For $p = 2s - 2$ the Lie algebra $\mathfrak{g}_{[0]_M}$ is isomorphic to $\mathfrak{gl}^{B_1}_{n_1} \times \mathfrak{gl}_{n_2}(\mathbb{C}) \times \cdots \times \mathfrak{gl}_{n_{s-1}}(\mathbb{C}) \times \mathfrak{gl}^{B_s}_{n_s}(\mathbb{C})$. Here either $B_1 = J_{n_1}$, $B_s = J_{n_s}$ or $B_1 = K_{n_1}$, $B_s = K_{n_s}$ depending on what

kind of Lie algebras is considered. It is clear that in the symplectic case the integers n_1 and n_s should be even. The Lie group G_0 is isomorphic to $\mathrm{GL}^{B_1}_{n_1}(\mathbb{C}) \times \mathrm{GL}_{n_2}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{n_{s-1}}(\mathbb{C}) \times \mathrm{GL}^{B_s}_{n_s}(\mathbb{C})$.

3.4 $\mathbb{Z}_M$-gradations of complex general linear Lie algebras of outer type

The consideration of this case is based on the observation that an arbitrary outer automorphism A of $\mathfrak{gl}_n(\mathbb{C})$ is conjugated to the automorphism defined by the relation

$$A(x) = -h\,{}^J x h^{-1},$$

where h is an element of $\mathrm{GL}_n(\mathbb{C}) \cap \mathrm{D}_n(\mathbb{C})$ such that ${}^J h h = I$. One can show that without loose of generality one can assume that $\det h = 1$. As in the orthogonal and symplectic cases $h^M = \nu I$, where ν is equal either to 1 or to -1. Furthermore, one can get convinced that outer type $\mathbb{Z}_M$-gradations of $\mathfrak{gl}_n(\mathbb{C})$ exist only for even M. We denote $M/2$ by N.

In the simplest case $h = I$, and there are only two nontrivial grading subspaces $\mathfrak{g}_{[0]_{2N}}$ and $\mathfrak{g}_{[N]_{2N}}$ whose elements are singled out by the conditions ${}^J x = -x$ and ${}^J x = x$, respectively. In this case, the Lie group G_0 coincides with $\mathrm{SO}_n(\mathbb{C})$ and one comes to the Toda equation (11) with $\varGamma$ being a mapping from $\mathcal{M}$ to $\mathrm{SO}_n(\mathbb{C})$, and C_+ and C_- being mappings from $\mathcal{M}$ to the space of $n \times n$ complex matrices x satisfying the equality ${}^J x = x$.

To analyze a general case one uses the following simple observations. Let B be an arbitrary nonsingular matrix. For any $h \in \mathrm{GL}_n(\mathbb{C})$ the mapping $A\colon \mathfrak{gl}_n(\mathbb{C}) \to \mathfrak{gl}_n(\mathbb{C})$ defined by the equality

$$A(x) = -h\,{}^B x h^{-1} \tag{17}$$

is an automorphism of $\mathfrak{gl}_n(\mathbb{C})$. By an inner automorphism of $\mathfrak{gl}_n(\mathbb{C})$ generated by an element $g \in \mathrm{GL}_n(\mathbb{C})$ the automorphism A is conjugated to an automorphism of the same form with the replacement $h \to g h g^{-1}$, $B \to {}^t(g^{-1}) B g^{-1}$. Note also that for any $g \in \mathrm{GL}_n(\mathbb{C})$ the replacement $h \to hg$, $B \to Bg$ does not change the automorphism A.

Since $h^{2N} = \nu I$ where ν is equal either to 1 or to -1, then the diagonal matrix elements of h are of modulus one. In the case when $\nu = 1$ it is convenient to represent each of them either as $\mathrm{e}^{2\pi \mathrm{i} m/2N}$ or as $-\mathrm{e}^{2\pi \mathrm{i} m/2N}$, where m is an integer satisfying the condition $0 < m \le N$. In the case when $\nu = -1$ we use either the representation $\mathrm{e}^{2\pi \mathrm{i}(m+1/2)/2N}$ or the representation $-\mathrm{e}^{2\pi \mathrm{i}(m+1/2)/2N}$ where m is again an integer satisfying the condition $0 < m \le N$. Multiplying the matrices h and B from the right by the appropriate diagonal matrix we exclude the second variant of the representation. Then using inner automorphisms of $\mathfrak{gl}_n(\mathbb{C})$ we order the diagonal elements of h in a convenient way. There are two different cases corresponding to two additional types of $\mathbb{Z}_M$-gradations of $\mathfrak{gl}_M(\mathbb{C})$.

Type II

Assume that there is no diagonal matrix elements of h equal to 1. In this case one succeeds in bringing h to the form (14) where $\mu_\alpha = \mathrm{e}^{2\pi \mathrm{i} m_\alpha/2N}$ if $\nu = 1$, or

$\mu_\alpha = \mathrm{e}^{2\pi\mathrm{i}(m_\alpha+1/2)/2N}$ if $\nu = -1$. In both cases the integers m_α satisfy the inequality $0 < m_1 < \ldots < m_p \leq N$ and the relations

$$m_{p-\alpha+1} = N - m_\alpha, \qquad \alpha = 1, \ldots, p.$$

The automorphism A generating the $\mathbb{Z}_M$-gradation under consideration is given now by the relation (17) where $B = K_n$. Introducing integers $k_\alpha = m_\alpha - m_{\alpha+1}$ one easily sees that the $\mathbb{Z}_{2N}$-gradation of $\mathfrak{gl}_n(\mathbb{C})$ under consideration is specified by the same data as a $\mathbb{Z}_N$-gradation of type I of the corresponding orthogonal or symplectic Lie algebra.

Type III

Let some diagonal matrix elements of h be equal to 1. In this case $\nu = 1$, and one brings h to the form (14) where $\mu_\alpha = \mathrm{e}^{2\pi\mathrm{i}m_\alpha/2N}$ with m_α satisfying the inequality $0 < m_1 < \ldots < m_p \leq N$ and the relations

$$m_{p-\alpha+2} = N - m_\alpha, \qquad \alpha = 2, \ldots, p.$$

The automorphism A generating the $\mathbb{Z}_M$-gradation under consideration is given now by the relation (17) where[6]

$$B = \begin{pmatrix} J_{n_1} & 0 \\ 0 & K_{n-n_1} \end{pmatrix}.$$

Here n_1 is the number of the diagonal matrix elements of h equal to 1. Introducing integers $k_\alpha = m_\alpha - m_{\alpha+1}$ one sees that the $\mathbb{Z}_{2N}$-gradation of $\mathfrak{gl}_n(\mathbb{C})$ under consideration is specified by the same data as a $\mathbb{Z}_N$-gradation of type II of the corresponding orthogonal or symplectic Lie algebra.

After all, one can show [17] that $\mathbb{Z}_{2N}$-gradations of $\mathfrak{gl}_n(\mathbb{C})$ of types II and III can be depicted by the scheme given in Figure 2. Here the pairs of elements of the ring $\mathbb{Z}_{2N}$ are the possible grading indices of the corresponding blocks in the block matrix representation (15) of a general element of $\mathfrak{gl}_n(\mathbb{C})$. The two possibilities are distinguished by the additional restrictions imposed on the blocks. Namely, when the grading index k is within the range $0 \leq k < N$ then $x_{\alpha\beta} = -({}^B x)_{\alpha\beta}$ for $\alpha \leq \beta$, and $x_{\alpha\beta} = ({}^B x)_{\alpha\beta}$ for $\alpha > \beta$, while for k from the range $N \leq k < 2N$ one has $x_{\alpha\beta} = ({}^B x)_{\alpha\beta}$ when $\alpha \leq \beta$, and $x_{\alpha\beta} = -({}^B x)_{\alpha\beta}$ if $\alpha > \beta$.

4 Explicit form of Toda equations associated with loop groups of complex classical Lie groups

In this section we describe the explicit form of the Toda equation associated with loop groups of the complex classical Lie groups. Actually we describe the explicit form of the Toda equation (9) which is equivalent to the genuine Toda equation (1) for the case of integrable $\mathbb{Z}$-gradations with finite dimensional grading subspaces.

Let G be a complex classical Lie group and $\mathfrak{g}$ its Lie algebra. Choose some $\mathbb{Z}_M$-gradation of $\mathfrak{g}$. As was demonstrated above, up to conjugations, any $\mathbb{Z}_M$-gradation of a complex classical Lie algebra has the structure depicted either by Figure 1 or by

[6]Note that in this case $n - n_1$ is even with necessity.

$$
\left(\begin{array}{c|c|c|c|c}
[0]_{2N} & [k_1]_{2N} & [k_1+k_2]_{2N} & \cdots & [\sum\limits_{\alpha=1}^{p-1} k_\alpha]_{2N} \\
[N]_{2N} & [k_1+N]_{2N} & [k_1+k_2+N]_{2N} & \cdots & [\sum\limits_{\alpha=1}^{p-1} k_\alpha + N]_{2N} \\
\hline
-[k_1]_{2N} & [0]_{2N} & [k_2]_{2N} & \cdots & \sum\limits_{\alpha=2}^{p-1} [k_\alpha]_{2N} \\
-[k_1+N]_{2N} & [N]_{2N} & [k_2+N]_{2N} & \cdots & [\sum\limits_{\alpha=2}^{p-1} k_\alpha + N]_{2N} \\
\hline
-[k_1+k_2]_{2N} & -[k_2]_{2N} & [0]_{2N} & \cdots & [\sum\limits_{\alpha=3}^{p-1} k_\alpha]_{2N} \\
-[k_1+k_2+N]_{2N} & -[k_2+N]_{2N} & [N]_{2N} & \cdots & [\sum\limits_{\alpha=3}^{p-1} k_\alpha + N]_{2N} \\
\hline
\vdots & \vdots & \vdots & \ddots & \vdots \\
\hline
-[\sum\limits_{\alpha=1}^{p-1} k_\alpha]_{2N} & -[\sum\limits_{\alpha=2}^{p-1} k_\alpha]_{2N} & -[\sum\limits_{\alpha=3}^{p-1} k_\alpha]_{2N} & \cdots & [0]_{2N} \\
-[\sum\limits_{\alpha=1}^{p-1} k_\alpha + N]_{2N} & -[\sum\limits_{\alpha=2}^{p-1} k_\alpha + N]_{2N} & -[\sum\limits_{\alpha=3}^{p-1} k_\alpha + N]_{2N} & \cdots & [N]_{2N}
\end{array}\right)
$$

Figure 2: The structure of an outer $\mathbb{Z}_{2N}$-gradation of $\mathfrak{gl}_n(\mathbb{C})$

Figure 2. In all the cases the Lie algebra $\mathfrak{g}_{[0]_M}$ and the Lie group are formed by block diagonal matrices, and one can parameterize the mapping γ as

$$
\gamma = \begin{pmatrix} \Gamma_1 & & & \\ & \Gamma_2 & & \\ & & \ddots & \\ & & & \Gamma_p \end{pmatrix}, \tag{18}
$$

where for each $\alpha = 1, \ldots, p$ the mapping Γ_α is a mapping from $\mathcal{M}$ to the Lie group $\mathrm{GL}_{n_\alpha}(\mathbb{C})$. In a general case the mappings Γ_α are not independent. They satisfy some restrictions imposed by the structure of the group G.

Let L be a positive integer such that the grading subspaces $\mathfrak{g}_{+[k]_M}$ and $\mathfrak{g}_{-[k]_M}$ for $0 < k < L$ are trivial. One can see that if $x \in \mathfrak{g}_{+[L]_M}$, then only the blocks $x_{\alpha,\alpha+1}$, $\alpha = 1, \ldots, p-1$, and x_{p1} in the block matrix representation (15) can be different from zero. Thus, the mapping c_+ has the structure given in Figure 3, where for each $\alpha = 1, \ldots, p-1$ the mapping $C_{+\alpha}$ is a mapping from $\mathcal{M}$ to the space of $n_\alpha \times n_{\alpha+1}$ complex matrices, and C_{+0} is a mapping from $\mathcal{M}$ to the space of $n_p \times n_1$ complex matrices. Here it is assumed that if some blocks among $x_{\alpha,\alpha+1}$, $\alpha = 1, \ldots, p-1$, and x_{p1} in the general block matrix representation (15) have the grading index different from $+[L]_M$, then the corresponding blocks in the block representation of c_+ are zero matrices.

Similarly, one can see that the mapping c_- has the structure given in Figure 4, where for each $\alpha = 1, \ldots, p-1$ the mapping $C_{-\alpha}$ is a mapping from $\mathcal{M}$ to the space of $n_{\alpha+1} \times n_\alpha$ complex matrices, and C_{-0} is a mapping from $\mathcal{M}$ to the space of $n_1 \times n_p$

$$\begin{pmatrix} 0 & C_{+1} & & & \\ & 0 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 0 & C_{+(p-1)} \\ C_{+0} & & & & 0 \end{pmatrix}$$

Figure 3: The structure of the mapping c_+

$$\begin{pmatrix} 0 & & & & C_{-0} \\ C_{-1} & 0 & & & \\ & \ddots & \ddots & & \\ & & \ddots & 0 & \\ & & & C_{-(p-1)} & 0 \end{pmatrix}$$

Figure 4: The structure of the mapping c_-

complex matrices. It is assumed that if some blocks among $x_{\alpha+1,\alpha}$, $\alpha = 1, \ldots, p-1$, and x_{1p} in the general block matrix representation (15) have the grading index different from $-[L]_M$, then the corresponding blocks in the block representation of c_- are zero matrices.

The conditions (10) imply

$$\partial_+ C_{-\alpha} = 0, \qquad \partial_- C_{+\alpha} = 0, \qquad \alpha = 0, \ldots, p-1. \tag{19}$$

Additionally, the mappings $C_{+\alpha}$ and $C_{-\alpha}$ should satisfy some restrictions imposed by the structure of the Lie algebra $\mathfrak{g}$.

Now assume that G is the Lie group $\mathrm{GL}_n(\mathbb{C})$ and we use a $\mathbb{Z}_M$-gradation of $\mathfrak{gl}_n(\mathbb{C})$ of type I. It is not difficult to show that in this case the Toda equation (9) for the mapping γ is equivalent to the following system of equations for the mappings Γ_α:

$$\begin{aligned} \partial_+ \left(\Gamma_1^{-1} \partial_- \Gamma_1\right) &= -\Gamma_1^{-1} C_{+1} \Gamma_2 C_{-1} + C_{-0} \Gamma_p^{-1} C_{+0} \Gamma_1, \\ \partial_+ \left(\Gamma_2^{-1} \partial_- \Gamma_2\right) &= -\Gamma_2^{-1} C_{+2} \Gamma_3 C_{-2} + C_{-1} \Gamma_1^{-1} C_{+1} \Gamma_2, \\ &\vdots \\ \partial_+ \left(\Gamma_{p-1}^{-1} \partial_- \Gamma_{p-1}\right) &= -\Gamma_{p-1}^{-1} C_{+(p-1)} \Gamma_p C_{-(p-1)} + C_{-(p-2)} \Gamma_{p-2}^{-1} C_{+(p-2)} \Gamma_{p-1}, \\ \partial_+ \left(\Gamma_p^{-1} \partial_- \Gamma_p\right) &= -\Gamma_p^{-1} C_{+p} \Gamma_1 C_{-p} + C_{-(p-1)} \Gamma_{p-1}^{-1} C_{+(p-1)} \Gamma_p. \end{aligned} \tag{20}$$

The Toda equations associated with loop groups of $\mathrm{SL}_n(\mathbb{C})$ in the case of $\mathbb{Z}_M$-gradations of inner type have actually the same form (20) as the Toda equations associated with loop groups of $\mathrm{GL}_n(\mathbb{C})$. Here the mappings Γ_α should satisfy the condition $\prod_{\alpha=1}^{p} \det \Gamma_\alpha = 1$. If there is a solution of a Toda equation associated with a loop group of $\mathrm{GL}_n(\mathbb{C})$ one can easily obtain a solution of the Toda equation associated with the corresponding loop group of $\mathrm{SL}_n(\mathbb{C})$. Every solution of a Toda equation associated with a loop group of $\mathrm{SL}_n(\mathbb{C})$ can be derived from a solution of the Toda equation associated with the corresponding loop group of $\mathrm{GL}_n(\mathbb{C})$.

Now let G be the Lie group $\mathrm{SO}_n(\mathbb{C})$ or the Lie group $\mathrm{Sp}_n(\mathbb{C})$ and we use a $\mathbb{Z}_M$-gradation of $\mathfrak{so}_n(\mathbb{C})$ or $\mathfrak{sp}_n(\mathbb{C})$ of type I. Here we have two different cases. For an even

$p = 2s$ the Toda equation (9) is equivalent to the system

$$\begin{aligned}
\partial_+ \left(\Gamma_1^{-1}\partial_-\Gamma_1\right) &= -\Gamma_1^{-1}C_{+1}\Gamma_2 C_{-1} + C_{-0}\,{}^J\Gamma_1 C_{+0}\Gamma_1, \\
\partial_+ \left(\Gamma_2^{-1}\partial_-\Gamma_2\right) &= -\Gamma_2^{-1}C_{+2}\Gamma_3 C_{-2} + C_{-1}\Gamma_1^{-1}C_{+1}\Gamma_2, \\
&\;\;\vdots \\
\partial_+ \left(\Gamma_{s-1}^{-1}\partial_-\Gamma_{s-1}\right) &= -\Gamma_{s-1}^{-1}C_{+(s-1)}\Gamma_s C_{-(s-1)} + C_{-(s-2)}\Gamma_{s-2}^{-1}C_{+(s-2)}\Gamma_{s-1}, \\
\partial_+ \left(\Gamma_s^{-1}\partial_-\Gamma_s\right) &= -\Gamma_s^{-1}C_{+s}\,{}^J(\Gamma_s^{-1})C_{-s} + C_{-(s-1)}\Gamma_{s-1}^{-1}C_{+(s-1)}\Gamma_s,
\end{aligned} \tag{21}$$

where $C_{+0} = -{}^JC_{+0}$, $C_{-0} = -{}^JC_{-0}$, and $C_{+s} = -{}^JC_{+s}$, $C_{-s} = -{}^JC_{-s}$ for the the orthogonal case, whereas $C_{+0} = {}^JC_{+0}$, $C_{-0} = {}^JC_{-0}$, and $C_{+s} = {}^JC_{+s}$, $C_{-s} = {}^JC_{-s}$ for the symplectic case. It should be noted that a $\mathbb{Z}_M$-gradation of $\mathfrak{gl}_n(\mathbb{C})$ of type II for $p = 2s$ also gives the equations (21). However, in this case additionally to ${}^JC_{+0} = -C_{+0}$ and ${}^JC_{-0} = -C_{-0}$, one has ${}^JC_{+s} = C_{+s}$ and ${}^JC_{-s} = C_{-s}$.

Returning to the case of $\mathbb{Z}_M$-gradations of $\mathfrak{so}_n(\mathbb{C})$ or $\mathfrak{sp}_n(\mathbb{C})$ of type I, one sees that for an odd $p = 2s-1$ the Toda equation (9) is equivalent to the system

$$\begin{aligned}
\partial_+ \left(\Gamma_1^{-1}\partial_-\Gamma_1\right) &= -\Gamma_1^{-1}C_{+1}\Gamma_2 C_{-1} + C_{-0}\,{}^J\Gamma_1 C_{+0}\Gamma_1, \\
\partial_+ \left(\Gamma_2^{-1}\partial_-\Gamma_2\right) &= -\Gamma_2^{-1}C_{+2}\Gamma_3 C_{-2} + C_{-1}\Gamma_1^{-1}C_{+1}\Gamma_2, \\
&\;\;\vdots \\
\partial_+ \left(\Gamma_{s-1}^{-1}\partial_-\Gamma_{s-1}\right) &= -\Gamma_{s-1}^{-1}C_{+(s-1)}\Gamma_s C_{-(s-1)} + C_{-(s-2)}\Gamma_{s-2}^{-1}C_{+(s-2)}\Gamma_{s-1}, \\
\partial_+ \left(\Gamma_s^{-1}\partial_-\Gamma_s\right) &= -{}^{B_s}(C_{-(s-1)}\Gamma_{s-1}^{-1}C_{+(s-1)}\Gamma_s) + C_{-(s-1)}\Gamma_{s-1}^{-1}C_{+(s-1)}\Gamma_s.
\end{aligned} \tag{22}$$

Here ${}^{B_s}\Gamma_s = \Gamma_s^{-1}$ with $B_s = J$ in the orthogonal case and $B_s = K$ in the symplectic case. One finds also that in the orthogonal case $C_{+s} = -{}^JC_{+s}$ and $C_{-s} = -{}^JC_{-s}$, while in the symplectic case one obtains $C_{+s} = {}^JC_{+s}$ and $C_{-s} = {}^JC_{-s}$. Using a $\mathbb{Z}_M$-gradation of $\mathfrak{gl}_n(\mathbb{C})$ of type II for $p = 2s-1$ one also comes to the equations (22) with $B_s = K$ and the conditions $C_{+s} = -{}^JC_{+s}$, $C_{-s} = -{}^JC_{-s}$. A $\mathbb{Z}_M$-gradation of $\mathfrak{gl}_n(\mathbb{C})$ of type III for $p = 2s-1$ gives the equations equivalent to the equations (22) with $B_s = J$ and $C_{+s} = {}^JC_{+s}$, $C_{-s} = {}^JC_{-s}$.

Finally, if we use $\mathbb{Z}_M$-gradations of $\mathfrak{so}_n(\mathbb{C})$ or $\mathfrak{sp}_n(\mathbb{C})$ of type III we see that the Toda equation (9) is equivalent to the system

$$\begin{aligned}
\partial_+ \left(\Gamma_1^{-1}\partial_-\Gamma_1\right) &= -\Gamma_1^{-1}C_{+1}\Gamma_2 C_{-1} + {}^{B_1}(\Gamma_1^{-1}C_{+1}\Gamma_2 C_{-1}), \\
\partial_+ \left(\Gamma_2^{-1}\partial_-\Gamma_2\right) &= -\Gamma_2^{-1}C_{+2}\Gamma_3 C_{-2} + C_{-1}\Gamma_1^{-1}C_{+1}\Gamma_2, \\
&\;\;\vdots \\
\partial_+ \left(\Gamma_{s-1}^{-1}\partial_-\Gamma_{s-1}\right) &= -\Gamma_{s-1}^{-1}C_{+(s-1)}\Gamma_s C_{-(s-1)} + C_{-(s-2)}\Gamma_{s-2}^{-1}C_{+(s-2)}\Gamma_{s-1}, \\
\partial_+ \left(\Gamma_s^{-1}\partial_-\Gamma_s\right) &= -{}^{B_s}(C_{-(s-1)}\Gamma_{s-1}^{-1}C_{+(s-1)}\Gamma_s) + C_{-(s-1)}\Gamma_{s-1}^{-1}C_{+(s-1)}\Gamma_s.
\end{aligned} \tag{23}$$

Here ${}^{B_1}\Gamma_1 = \Gamma_1^{-1}$, ${}^{B_s}\Gamma_s = \Gamma_s^{-1}$ with $B_1 = J$, $B_s = J$ in the orthogonal case and $B_1 = K$, $B_s = K$ in the symplectic case. A $\mathbb{Z}_M$-gradation of $\mathfrak{gl}_n(\mathbb{C})$ of type III for $p = 2s-2$ gives the equations (23) with $B_1 = J$, $B_s = K$.

It is worth noting for each type of the Toda systems considered above that if for some α one has $C_{+\alpha} = 0$ or $C_{-\alpha} = 0$, then the considered system of equations becomes a Toda system associated with the corresponding finite dimensional Lie group, see, for example, the papers [13, 14]. Hence, to have equations which are really associated with loop groups one must assume that all mappings $C_{+\alpha}$ and $C_{-\alpha}$ are nontrivial. This is possible only when $k_\alpha = L$ for each $\alpha = 1, \ldots, p-1$ and $M = pL$. Without any loss of generality one can assume that $L = 1$.

The equations (21), (22) and (23) can be obtained from the equations (20) by some folding procedure described in Appendix A. That procedure also helps to understand that there may be only the described classes of Toda equations associated with the loop groups under consideration.

5 Simplest case: Non-abelian sinh-Gordon and sine-Gordon equations

Defining the mappings $\varGamma_\alpha$, $C_{+\alpha}$ and $C_{-\alpha}$ for all integer values of the index α with the periodicity conditions $\varGamma_{\alpha+p} = \varGamma_\alpha$, $C_{-(\alpha+p)} = C_{-\alpha}$ and $C_{+(\alpha+p)} = C_{+\alpha}$, one can treat the system (20) as the infinite periodic system

$$\partial_+(\varGamma_\alpha^{-1}\partial_-\varGamma_\alpha) = -\varGamma_\alpha^{-1}C_{+\alpha}\varGamma_{\alpha+1}C_{-\alpha} + C_{-(\alpha-1)}\varGamma_{\alpha-1}^{-1}C_{+(\alpha-1)}\varGamma_\alpha.$$

In particular, when $n = rp$ and $n_\alpha = r$, $C_{-\alpha} = C_{+\alpha} = I_r$ one comes to the equations [23, 24]

$$\partial_+(\varGamma_\alpha^{-1}\partial_-\varGamma_\alpha) = -\varGamma_\alpha^{-1}\varGamma_{\alpha+1} + \varGamma_{\alpha-1}^{-1}\varGamma_\alpha.$$

Putting $p = 2$ one finds the simplest set of Toda equations

$$\partial_+(\varGamma_1^{-1}\partial_-\varGamma_1) = -\varGamma_1^{-1}\varGamma_2 + \varGamma_2^{-1}\varGamma_1, \qquad \partial_+(\varGamma_2^{-1}\partial_-\varGamma_2) = -\varGamma_2^{-1}\varGamma_1 + \varGamma_1^{-1}\varGamma_2,$$

where $\varGamma_1$ and $\varGamma_2$ are $r \times r$ matrices. These equations are invariant with respect to the transformations $\varGamma_1 \to {}^t(\varGamma_2^{-1})$, $\varGamma_2 \to {}^t(\varGamma_1^{-1})$, where the superscript t means the usual transposition. This invariance implies the possibility of the reduction to the case when $\varGamma_1 = {}^t(\varGamma_2^{-1}) = \varGamma$. Here the equations under consideration are reduced to the equation

$$\partial_+(\varGamma^{-1}\partial_-\varGamma) = -({}^t\varGamma\varGamma)^{-1} + {}^t\varGamma\varGamma. \tag{24}$$

For the matter of possible physical applications, it is interesting to consider real forms of Toda equations (9). Let an involutive antiholomorphic automorphism σ of G and its Lie algebra counterpart, an involutive antilinear automorphism Σ of $\mathfrak{g}$ consistent with the $\mathbb{Z}$-gradation, be given. The latter means that if $x \in \mathfrak{g}_{[k]_M}$, then $\Sigma(x) \in \mathfrak{g}_{[k]_M}$. Suppose also that $\Sigma(c_+) = c_+$ and $\Sigma(c_-) = c_-$. In this case if γ is a solution to the Toda equation (9), then $\sigma \circ \gamma$ is also a solution to this equation,[7] and consistent reduction to the case when $\sigma \circ \gamma = \gamma$ is possible.

One can construct two non-equivalent real forms of the Toda equation (24). The first one is based on the non-compact real form given by real matrices. In this case $\sigma(\varGamma) = \varGamma^* = \varGamma$. For $r = 1$ putting here $\varGamma = \exp(F)$ one obtains the sinh-Gordon equation

$$\partial_+\partial_-F = 2\sinh F.$$

[7] Here we assume that $\mathcal{M}$ is $\mathbb{R}^2$.

Another one is based on the compact real form given by unitary matrices. In this case $\sigma(\mathit{\Gamma}) = (\mathit{\Gamma}^{\dagger})^{-1} = \mathit{\Gamma}$ and it is convenient to write the Toda equation (24) in the form

$$\partial_{+}(\mathit{\Gamma}^{-1}\partial_{-}\mathit{\Gamma}) = -({}^{t}\mathit{\Gamma}\mathit{\Gamma})^{\dagger} + {}^{t}\mathit{\Gamma}\mathit{\Gamma}.$$

For $r = 1$ putting $\mathit{\Gamma} = \exp(\mathrm{i}F)$ one obtains the sine-Gordon equation

$$\partial_{+}\partial_{-}F = 2\sin F.$$

6 Conclusions

It is observed that there are four non-equivalent classes of Toda equations associated with loop groups of complex classical Lie groups, when their Lie algebras are endowed with integrable $\mathbb{Z}$-gradations with finite dimensional grading subspaces. The first class is presented by the equations (20). It is the basic class of the loop Toda equations with no restrictions on the mappings $\mathit{\Gamma}_{\alpha}$ and $C_{\pm\alpha}$.

In contrast, the other three classes of loop Toda equations, presented here by the equations (21), (22) and (23), arise from the equations (20) when one imposes some restrictions on the mappings $\mathit{\Gamma}_{\alpha}$ and $C_{\pm\alpha}$ coming from the specific group and algebra defining conditions and also effected by the gradation type. It is interesting, for example, that all these three classes of loop Toda equations are revealed for the general linear case too, under the outer type gradations, although there was no specifying group and algebra condition at the beginning. The classification results are supplied further with a graphic folding procedure, see the Appendix A.

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A Graphic representation for loop Toda equations

Here a graphic representation for the Toda equations associated with loop groups is described. This representation can be useful for a simple understanding that only four non-equivalent classes of loop Toda equations may actually arise.

We use the following identification rules illustrated by Figure 5. Associate every mapping $\mathit{\Gamma}_{\alpha}$ entering the relation (18) to a small disk on a circle, and the mappings $C_{\pm\alpha}$ to the arc between two such disks already identified with $\mathit{\Gamma}_{\alpha}$ and $\mathit{\Gamma}_{\alpha+1}$. The numbering in α is along the circle in the anti-clockwise direction. The whole Toda system corresponds to a circle with p small disks attached.

For the loop groups of the general linear Lie groups, when $\mathbb{Z}_{M}$-gradations are of inner type, one does not have any additional restrictions to the mappings γ, c_{+} and c_{-}, apart from their canonical forms given by the relation (18) and Figures 3 and 4. In this case, one is left with the simple picture given in Figure 5 depicting the Toda equations (20).

Essentially different situations are revealed for the cases of other classical Lie groups under inner and outer type $\mathbb{Z}_{M}$-gradations and the general linear Lie group under outer type $\mathbb{Z}_{M}$-gradations.

The application of the group and algebra defining conditions to the mappings γ and c_{+}, c_{-}, respectively, makes one fold up the correspondingly marked circle with

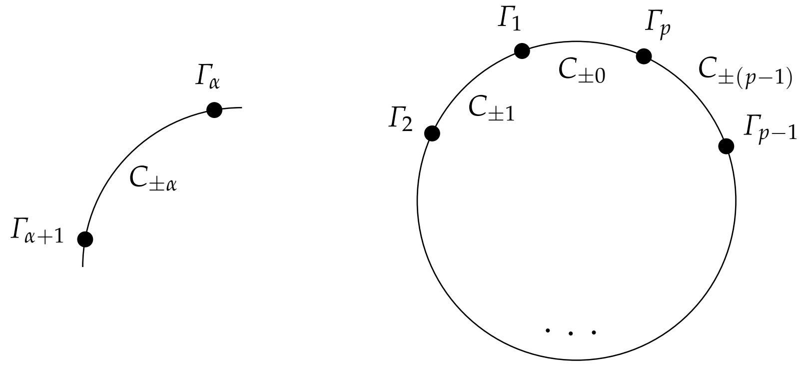

Figure 5: Identification rules and the general linear case under inner type $\mathbb{Z}_M$-gradations: The Toda equations (20).

respect to its diameter, the dash line in our pictures. A Toda system is consistent with its graphic representation, if each object on the circle identified with Γ_α and $C_{\pm\alpha}$, $\alpha = 1, \dots, p$, finds its image counterpart on the other side under such a folding.

It is clear that there are only three principally different possibilities to arrange a picture for all partitions: two for an even p, Figures 6 and 7, and one for an odd p,

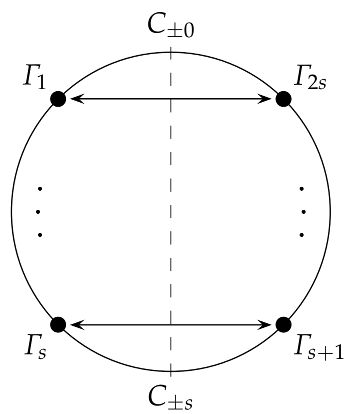

Figure 6: An even number $p = 2s$. The Toda equations (21) where ${}^J C_{\pm 0} = \varepsilon C_{\pm 0}$ and ${}^J C_{\pm s} = \varepsilon C_{\pm s}$.

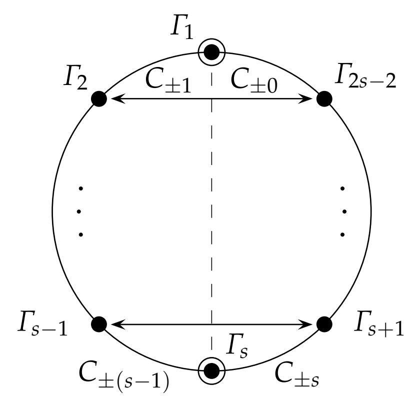

Figure 7: An even number $p = 2s - 2$. The Toda equations (23.) where ${}^{B_1}\Gamma_1 = \Gamma_1^{-1}$ and ${}^{B_s}\Gamma_s = \Gamma_s^{-1}$

the left picture in Figure 8. The additionally encircled small disks are ones which are folded up with themselves. The right picture in Figure 8 describes another variant of folding. It corresponds to the Toda equations which can be obtained from the equations (22) by the substitution $\Gamma_\alpha \to {}^{B_s}(\Gamma_{s-\alpha+1}^{-1})$, $C_{\pm\alpha} \to {}^J C_{\pm(s-\alpha)}$ supplemented by the change $B_s \to B_1$.

Finally, it is interesting to make a comparison with the Toda systems associated with finite dimensional Lie groups [13, 14]. Considering a similar graphic representation on a line for such Toda systems and taking into account that no periodicity is at hand, one can see that only two different classes of Toda systems, one for an even and one for an odd number p, associated with the orthogonal and symplectic Lie groups can arise there in addition to the general linear case.

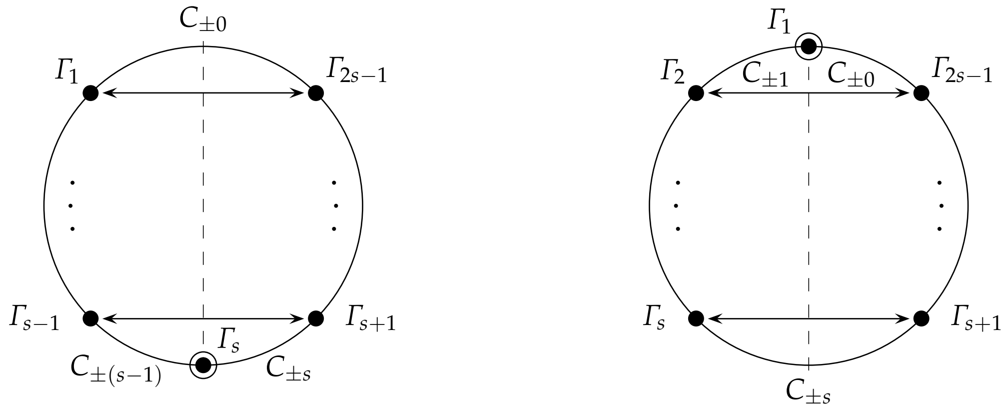

Figure 8: An odd number $p = 2s - 1$. The Toda equations (22), where ${}^{B_s}\Gamma_s = \Gamma_s^{-1}$ and ${}^{J}C_{\pm 0} = \varepsilon C_{\pm 0}$, and the equivalent Toda equations with ${}^{B_1}\Gamma_1 = \Gamma_1^{-1}$ and ${}^{J}C_{\pm s} = \varepsilon C_{\pm s}$. Here ε is either $+$ or $-$.

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